Problem Set 9

What problems do we think can't be solved efficiently? Are they purely theoretical concepts invented to scare mathematicians, or might you encounter it some day?

Start this problem set early. It contains three problems (plus one survey question and one extra credit problem). I suggest reading through this problem set at least once as soon as you get it to get a sense of what it covers.

As much as you possibly can, please try to work on this problem set individually. That said, if you do work with others, please be sure to cite who you are working with and on what problems. For more details, see the section on the honor code in the course information handout.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. You should, at the very least, state what type of proof you are using, and (if proceeding by contradiction, contrapositive, or induction) state exactly what it is that you are trying to show. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

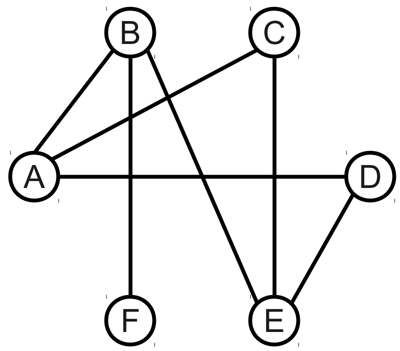
This problem set has 125 possible points. It is weighted at 6% of your total grade.

Good luck, and have fun!

Due Wednesday, June 6th at 2:15 PM

Problem One: The Long Path Problem (30 points)

Given an undirected graph $G = (V, E)$, a *simple path* in a G is a path between two nodes $u, v \in V$ such that no node is repeated on the path. For example, given this graph:



A → C → E is a simple path from A to E, but A → B → E → C → A → D is not a simple path because node A is visited twice.*

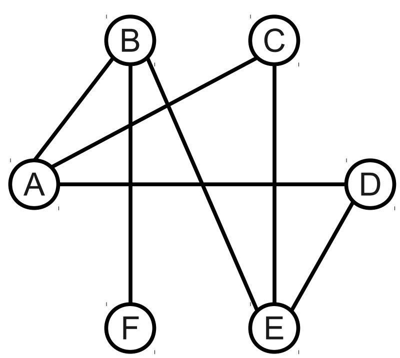
Consider the following language:

$ULONGPATH$ = { ⟨G, u, v, k⟩ | G is an undirected graph,
u and v are nodes in the graph, and
there exists a simple path from u to v containing k nodes. }

For example, if G is the above graph, then ⟨G, D, F, 6⟩ ∈ $ULONGPATH$ because there is a simple path of six nodes from D to F (namely, D → A → C → E → B → F), but ⟨G, A, C, 5⟩ ∉ $ULONGPATH$ because there is no simple path of 5 nodes from A to C.

i. Show that $ULONGPATH$ is in **NP** by designing a deterministic polynomial-time verifier for it. You should prove that your verifier is correct by showing that ⟨G, u, v, k⟩ ∈ $ULONGPATH$ iff there is some x such that your verifier accepts ⟨G, u, v, k, x⟩. However, you can give an informal justification as to why your machine runs in polynomial time.

In an undirected graph $G = (V, E)$, a *Hamiltonian path* is a simple path between two nodes u and v that visits every node in the graph exactly once. For example, in this graph:



The path F → B → E → C → A → D is a Hamiltonian path from F to D, but F → B → E → D is not (because it doesn't visit every node), nor is F → B → E → D → A → C → E → D (because it is not a simple path).

* Although we defined a path in a graph as a series of edges, it is often easier to reason about the path as the series of nodes that it passes through. Throughout this problem set, we will adopt this convention.

The language *UHAMPATH* is defined as follows:

$$UHAMPATH = \{\langle G, u, v\rangle \mid G \text{ is an undirected graph and there is a Hamiltonian path from } u \text{ to } v.\}$$

UHAMPATH is known to be **NP**-complete by a fairly clever series of reductions from SAT; see Sipser, page 291 for more details.

ii. Prove that *ULONGPATH* is **NP**-complete by showing that $UHAMPATH \leq_P ULONGPATH$. You should prove your reduction is correct (i.e, $w \in UHAMPATH$ iff $f(w) \in ULONGPATH$), but feel free to justify informally why your reduction works in polynomial time.

Problem Two: P $\stackrel{?}{=}$ NP (30 points)

This problem explores the question

What would it take to prove whether or not P = NP?

Below are twelve numbered statements. For each statement, identify whether the statement would definitely prove that **P** = **NP**, definitely prove that **P** ≠ **NP**, or not prove either result. No justification is necessary.

(1) There is an **NP** language that can be decided in polynomial time.

(2) There is an **NP-*complete*** language that can be decided in polynomial time.

(3) There is an **NP-*hard*** language that can be decided in polynomial time.

(4) There is an **NP** language that cannot be decided in polynomial time.

(5) There is an **NP-*complete*** language that cannot be decided in polynomial time.

(6) There is an **NP-*hard*** language that cannot be decided in polynomial time.

(7) There is ***some*** **NP**-complete language that can be decided in $O(2^n)$ time.

(8) There is ***no*** **NP**-complete language that can be decided in $O(2^n)$ time.

(9) There is a polynomial-time ***verifier*** for every language in **NP**.

(10) There is a a polynomial-time ***decider*** for every language in **NP**.

(11) All languages in **NP** are decidable.

(12) All languages in **P** are decidable.

Problem Three: The Big Picture (60 points)

We have covered a *lot* of ground in this course throughout our whirlwind tour of computability and complexity theory. This last question surveys what we have covered so far by asking you to see how everything we have covered relates.

Take a minute to review the hierarchy of languages we set up:

$$\mathbf{REG} \subset \mathbf{DCFL} \subset \mathbf{CFL} \subset \mathbf{P} \subseteq \mathbf{NP} \subset \mathbf{R} \subset \mathbf{RE} \subset \mathbf{ALL}$$

The following questions ask you to provide examples of languages at different spots within this hierarchy. In each case, you should provide an example of a language, but you don't need to formally prove that it has the properties required. Instead, describe a proof technique you could use to show that the language has the required properties. There are many correct answers to these problems, and we'll accept any of them.

i. Give an example of a regular language. How might you prove that it is regular?

ii. Give an example of a context-free language is not regular. How might you prove that it is context-free? How might you prove that it is not regular?

iii. Give an example of a language in **P** that is not context-free. How might you prove that it is in **P**? How might you prove that it is not context-free?

iv. Give an example of a language in **NP** suspected not to be in **P**. How might you prove that it is in **NP**? Why do we think that it is not contained in **P**?

v. Give an example of a language in **RE** not contained in **R**. How might you prove that it is **RE**? How might you prove that it is not contained in **R**?

vi. Give an example of a language in co-**RE** not contained in **R**. How might you prove that it is co-**RE**? How might you prove that it is not contained in **R**?

vii. Give an example of a language that is neither **RE** nor co-**RE**. How might you prove it is not contained in **RE**? How might you prove it is not contained in co-**RE**?

Problem Four: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. **If we should keep any one thing about this course the same in future offerings, what would it be?**

ii. **If you could change any one thing about this course, what would it be?**

iii. **What topic did you think was the most interesting? What topic did you think was the least interesting?**

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-spr1112-submissions@lists.stanford.edu) with the string "[PS9]" somewhere in the subject line. If you do submit electronically, please submit your assignment as a single PDF if at all possible. Sending multiple image files makes it much harder to grade your submission.

If you are an SCPD student, we would strongly prefer that you submit solutions via email. Please contact us if this will be a problem.

Extra Credit Problem (Worth an automatic A+, $1,000,000, and a Stanford Ph.D)

Prove or disprove: **P** = **NP**.